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COMMENT

About diffusion processes in disordered systems

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Abstract. Different diffusion processes can be defined on random networks like the infinite incipient clusters at percolation threshold. The long-time behaviour of two such processes is shown to be the same. In particular the mean-square displacements and the autocorrelation function scale with the same exponents in the two cases.

Diffusion problems on random networks have attracted much attention in the last decade. Since de Gennes (1976) introduced the problem of a particle (the ant) moving randomly on a random environment (the labyrinth) some progress has been made (Straley 1980, Alexander and Orbach 1982, Rammal and Toulouse 1983, Rammal *et al* 1985, de Arcangelis *et al* 1985). In particular scaling laws have been conjectured for diffusion processes on statistical or deterministic self-similar structures (fractals) (Alexander and Orbach 1982, Rammal and Toulouse 1983).

However, different diffusion problems can be defined on structures with inhomogeneous coordination depending on the jumping probabilities from a site to one of its nearest neighbours (NN) (even if simple generalisations should be easy to deal with, we shall not consider them here).

We shall be interested in two types of random walks nicknamed the myopic and blind ant problems (Mitescu and Roussenq 1983; see also Majid *et al* 1984, Havlin and Ben-Avraham 1987).

Since we are now accustomed to find surprises for critical phenomena on fractal lattices with respect to those on regular lattices (Rammal *et al* 1985, de Arcangelis *et al* 1985) it is legitimate to suspect that the myopic and blind ants could have different scaling behaviours at long times.

There is numerical evidence that the two ants moving on the infinite incipient percolation cluster in two dimensions lie on the same universality class (Majid *et al* 1984). The goal of this paper is to prove, in all generality, that the myopic and blind ant define the same critical exponents at large time.

The proof is not difficult and is based on the formal solution for the probability distribution in terms of a sum over a set of walks. The main ingredient is that this set of walks is the same for the two types of ant while the statistical weight of one ant can be bounded from below and from above by the statistical weight of the other ant. To be definite let us work with the discrete-time version of the master equation obeyed by the probability distribution $p_x(n)$ that the ant is at site x at 'time' n

$$p_x(n+1)=p_x(n)+\sum_y (w_{xy}p_y(n)-w_{yx}p_x(n)). \qquad (1)$$

w_{xy} is the probability to jump from the site y to the site x and it is different from zero only when x and y are NN sites of the network. If the network is random we have also to average $p_x(n)$ with a network-dependent statistical weight.

In the following we shall work with the generating function of $p_x(n)$ (discrete Laplace transform) defined as

$$P_x(\omega) = \sum_{n=0}^{\infty} \frac{p_x(n)}{(1+\omega)^n} \qquad \omega > 0. \tag{2}$$

If the ant is at x_0 at $n=0$, i.e. $p_x(n=0) = \delta_{xx_0}$ then (1) in terms of $P_x(\omega)$ becomes

$$\frac{\omega}{1+\omega} P_x(\omega) = \frac{1}{1+\omega} \sum_y (w_{xy} P_y(\omega) - w_{yx} P_x(\omega)) + \delta_{xx_0} \tag{3}$$

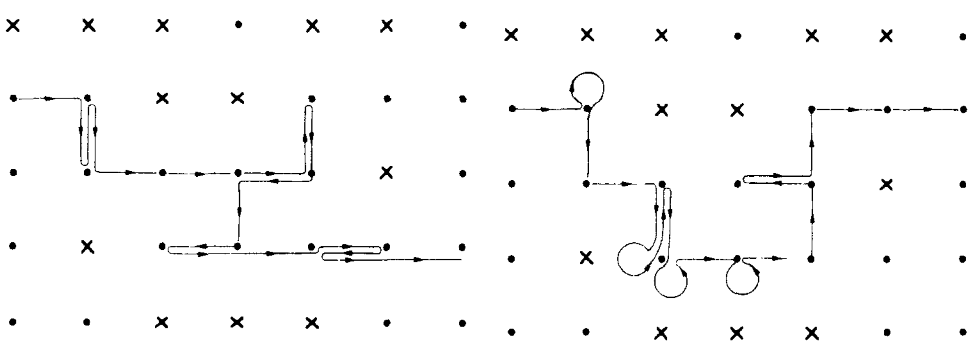
where the dependence of $P_x(\omega)$ on x_0 is understood. The myopic and blind ant (Mitescu and Roussenq 1983) have the following jumping probabilities†:

$$w_{xy} = 1/z_y \tag{4a}$$

$$w_{xy} = 1/z \tag{4b}$$

respectively where z_y is the coordination number of site y, i.e. the number of NN sites of y belonging to the network, while z is the coordination number of each site of the lattice where the network is embedded.

In other words the mypoic ant, defined by (1) and (4a), moves in one of the NN sites of its present position at each time step. The blind ant at each time step tries to jump on one of all possible NN sites (z in our case): if the site belongs to the network then it will be the new position of the ant otherwise the ant remains where it was. Figure 1 shows possible walks for the two types of ants.



(*a*) (*b*)

Figure 1. Subsets of a lattice where the ants can move only between nearest-neighbour sites which are available (•). Possible 18-step walks for the myopic and blind ant are shown in (*a*) and (*b*), respectively. In counting the steps for the blind ant one has also to take into account for the trials (represented by closed loops) to go to non-available sites (×).

† More generally we could consider all w_{xy} multiplied by a constant factor w. Looking at (3) one easily recognises that the new solution can be expressed in terms of the old one as $[(1+\omega)/(w+\omega)]P_x(\omega/w)$.

To show the equivalence of the two problems in the large-time limit it is convenient to introduce the following definitions:

$$w_x = 1 - \sum_y w_{yx} \tag{5}$$

$$G_x(\omega) = \left(1 - \frac{w_x}{1+\omega}\right) P_x(\omega). \tag{6}$$

Equation (3) can be rewritten as

$$G_x(\omega) = \sum_y \frac{w_{xy}}{1+\omega-w_y} G_y(\omega) + \delta_{xx_0}. \tag{7}$$

The (formal) solution of (7) can be expressed as a weighted sum over oriented walks W joining the initial and final sites x_0 and x respectively. If we represent an n-step walk W as the sequence of visited sites $(x_0, x_1, x_2, \ldots, x_n \equiv x)$ where x_i and x_{i+1} are NN and belong to the network† then the weight associated to W is

$$\mathscr{P}_{\mathrm{W}}(\omega) = \prod_{i=0}^{n-1} \frac{w_{x_{i+1}x_i}}{1+\omega-w_{x_i}} \tag{8}$$

and

$$G_x(\omega) = \delta_{xx_0} + \sum_{\mathrm{W}} \mathscr{P}_{\mathrm{W}}(\omega). \tag{9}$$

For the diffusion processes under consideration we have

$$\mathscr{P}^{(\mathrm{m})}_{\mathrm{W}}(\omega) = \prod_{i=0}^{n-1} [z_{x_i}(1+\omega)]^{-1} \tag{10a}$$

and

$$\mathscr{P}^{(\mathrm{b})}_{\mathrm{W}}(\omega) = \prod_{i=0}^{n-1} \left[z_{x_i}\left(1 + \frac{z}{z_{x_i}}\omega\right)\right]^{-1} \tag{10b}$$

where, from now on, the superscript will distinguish between myopic (m) and blind (b) ants. The set of walks, over which the sum in (9) runs, is the same for the two ants. Indeed it is just the set of walks of the myopic ant.

Apart from the trivial case in which the initial site is not surrounded by available sites ($z_{x_0} = 0$ implies $p_x(n) = \delta_{xx_0}$ independently of n) we have $1 \leq z_x \leq z$ for each site visited by the walk.

Thus the following inequalities are an immediate consequence of (10):

$$\mathscr{P}^{(\mathrm{m})}_{\mathrm{W}}(z\omega) \leq \mathscr{P}^{(\mathrm{b})}_{\mathrm{W}}(\omega) \leq \mathscr{P}^{(\mathrm{m})}_{\mathrm{W}}(\omega) \tag{11}$$

which imply analogous inequalities for the G_x while, using (5) and (6), we obtain for the P_x

$$P^{(\mathrm{m})}_x(z\omega) \leq P^{(\mathrm{b})}_x(\omega) \leq z\frac{1+\omega}{1+z\omega} P^{(\mathrm{m})}_x(\omega). \tag{12}$$

Let us define the mean-square displacement as

$$\langle R^2(\omega)\rangle^{(i)} = \sum_x (x-x_0)^2 P^{(i)}_x(\omega) \frac{\omega}{1+\omega} \tag{13}$$

† Of course, a given site can appear more than one time in the sequence.

where i denotes m or b and the last factor takes into account the normalisation condition

$$\sum_x P_x^{(i)}(\omega) = \frac{\omega + 1}{\omega} \tag{14}$$

which is a consequence of (3). From inequalities (12) and equation (13) it follows that

$$\langle R^2(z\omega)\rangle^{(\mathrm{m})} \leqslant z\langle R^2(\omega)\rangle^{(\mathrm{b})} \leqslant z^2 \frac{1+\omega}{1+z\omega} \langle R^2(\omega)\rangle^{(\mathrm{m})} \tag{15}$$

which holds in general for the average of any positive function f_x. Of course (15) will be valid also after averaging over the random network and over the initial site x_0; we shall not introduce new notations for these last averages and from now on they will be understood. If we assume scaling in the standard fashion, i.e. at low ω, the mean-square displacement and autocorrelation function behave as

$$\langle R^2(\omega)\rangle^{(i)} \sim \omega^{-2\nu^{(i)}} \tag{16}$$

and

$$P_{x_0}^{(i)}(\omega) \sim \omega^{d_s^{(i)}/2-1} \tag{17}$$

respectively†, then from (15) and (12) one obtains

$$\nu^{(\mathrm{m})} = \nu^{(\mathrm{b})} \equiv \nu \qquad \text{and} \qquad d_s^{(\mathrm{m})} = d_s^{(\mathrm{b})} \equiv d_s. \tag{18}$$

Thus the myopic and blind ants have the same asymptotic behaviours. More generally one could foresee the possibility that in (16) and (17) there appear some power of $|\ln \omega|$ to multiply the leading behaviours. However, inequalities (12) and (15) would imply that these corrections are also the same for the two diffusion processes. This concludes the proof.

The exponent ν is known as the inverse of the walk fractal dimension d_w while d_s has been called fracton or spectral dimension (Alexander and Orbach 1982, Rammal and Toulouse 1983) and, in general, it is different from the fractal dimension d_f of the structure on which the diffusion takes place. These exponents are not all independent but $2d_f/d_w = d_s$ (Alexander and Orbach 1982) which is an intrinsic property of the fractal structure.

Summarising, we proved that two diffusive processes on a fractal, known as the myopic and blind ant in the labyrinth (Mitescu and Roussenq 1983; see also Majid *et al* 1984) define asymptotically the same set of critical indices.

We should stress, however, that at variance from what occurs on lattices with uniform coordination, i.e. z_x independent on x, all walks of the same length do not have the same statistical weight (see equations (10)). One can show on specific examples (Maritan 1987) that the set of all walks of a given length n with each walk equally weighted in the averages defines a new set of exponents characterising the entropy and the mean end-to-end distance of the walks in the large-n limit.

After this paper was completed the authors became aware of a preprint by Harris *et al* (1987) in which the same problem has been discussed. Among other results it is proved under rather plausible assumptions that the two ants have the same asymptotic behaviour.

† In terms of discrete time, (16) and (17) mean that at large n $\langle R^2(n)\rangle^{(i)} = \Sigma_x (x - x_0)^2 p_x^{(i)}(n) \sim n^{2\nu^{(i)}}$ and $p_{x_0}^{(i)}(n) \sim n^{-d_s^{(i)}/2}$ respectively.

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